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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM

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SOME NEW PROBLEMS ON SHELLS AND THIN STRUCTURES

By V. S. Vlasov

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1. Cylindrical shells of arbitrary section, reinforced by longitudinal and transverse members (stringers and ribs) are considered by us, for a sufficiently close spacing of the ribs, as in our previously published papers (references 1 and 2), as thin-walled orthotropic spatial systems at the cross-sections of which only axial (normal and shearing) forces can arise. The longitudinal bending and twisting moments, due to their weak effect on the stress state of the shell, are taken equal to zero. Along the longitudinal sections of the shell there may arise transverse forces in addition to the normal and shearing forces. Under the so-called static assumptions there is taken for the computation model of the shell a thin-walled spatial system consisting along its length (along a generator) of an infinite number of elementary strips capable of bending. Each of these strips is likened to a curved rod operating in each of its sections not only in tension (compression) but also in transverse bending and shear. The interaction between two adjoining transverse strips in the shell expresses itself in the transmission from one strip to the other of only the normal and shearing stresses.

The static structure of the computation model here described is shown in figure 1, where the connections through which the normal and shearing stresses transmitted from one transverse strip to another are indicated schematically by the rods located in the middle surface of the shell.

In addition to the static hypothesis we introduce also geometric hypotheses. According to the latter the elongational deformations of the shell along lines parallel to the generator of its middle surface and the shear deformations in the middle surface, as magnitudes having little effect on the state of the fundamental internal forces of the shell, are taken equal to zero. The deformations of the shell in our computational model are such that in the first place the lines of this surface perpendicular to the generator are inextensible at each point and in the second place the angles between the lines of principal curvature (the coordinate lines) which are straight before the deformation remain straight after the deformation.

The differential equations of equilibrium of the cylindrical shell, by virtue of the statical hypotheses assumed, will have the following form (figs. 2 and 3).

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*"Nekotórie Novie Zadachi Stroitelnoi Mekhaniki Obolochek i Tonkostennikh Konstruktsii." Izvestia Akademii Nauk, 1947, No. 1, pp. 27-53.

$$\frac{\partial T_1}{\partial z} + \frac{\partial S}{\partial s} + p_z = 0$$

$$\frac{\partial N}{\partial s} + \frac{T_2}{R} + p_n = 0$$

$$\frac{\partial T_2}{\partial s} + \frac{\partial S}{\partial z} - \frac{N}{R} + p_s = 0 \qquad (1.1)$$

$$N - \frac{\partial G}{\partial s} = 0$$

where z and s are the coordinates of a point of the middle surface, z being the distance along the generator and s, along the direction at right angles; $R = R(s)$ is the radius of curvature, p_z, p_s, and p_n, the components of the vector of the surface load.

The system of equations (1.1) by elimination of the forces S, T_2, and N is reduced to a single equation

$$\frac{\partial^2(\sigma\delta)}{\partial z^2} + \Omega G = P \qquad (1.2)$$

where $\sigma = \frac{T_1}{\delta}$ is the axial normal stress (δ is the thickness of the shell), P is a function depending on the components of the external surface load and determined by the equation

$$P = -\frac{\partial p_z}{\partial z} + \frac{\partial p_s}{\partial s} - \frac{\partial^2}{\partial s^2}(Rp_n) \qquad (1.3)$$

In equation (1.2) Ω denotes the differential operator of the fourth order in the variable s. This operator, as shown in reference 5, is connected with the "law of sectorial areas" and has the form

$$\Omega = \frac{\partial^2}{\partial s^2}\left(R\frac{\partial^2}{\partial s^2}\right) + \frac{\partial}{\partial s}\left(\frac{1}{R}\frac{\partial}{\partial s}\right) \tag{1.4}$$

If there are no surface forces the required internal forces of the shell can be expressed in terms of the single function $\Phi = \Phi(z,s)$ by the equations

$$\left.\begin{aligned} T_1 &= \sigma\delta = \Omega\Phi \\ S &= -\Omega_1\frac{\partial^2\Phi}{\partial z^2} \\ T_2 &= R\frac{\partial^4\Phi}{\partial z^2\partial s^2} \\ G &= -\frac{\partial^2\Phi}{\partial z^2} \\ N &= -\frac{\partial^3\Phi}{\partial s\,\partial z^2} \end{aligned}\right\} \tag{1.5}$$

in which Ω_1 is the differential operator of the third order also in the variable s

$$\Omega_1 = \frac{\partial}{\partial s}\left(R\frac{\partial^2}{\partial s^2}\right) + \frac{1}{R}\frac{\partial}{\partial s} \tag{1.6}$$

and is connected with the operator Ω by the relation

$$\Omega\Phi = \frac{\partial}{\partial s}(\Omega_1\Phi) \tag{1.7}$$

For $R\frac{\partial^2 \Phi}{\partial s^2} = \varphi$ and $R \to \infty$ the first three relations in equations (1.5) go over into the well-known equations of Airy for the plane problem of the theory of elasticity. The function Φ may be called the stress function for the cylindrical shell. This function, as in the plane problem, in the computational model assumed by us, is conditioned only by the static hypotheses and plays the part of the fundamental static undetermined magnitude.

If the surface load is different from zero there must be added to the right side of equations (1.5) the corresponding particular solutions of the nonhomogeneous static equations. These particular solutions may be obtained from equations (1.1) on the assumption that $G = 0$ or $T_1 = 0$. In the second case $\left(\text{for } T_1 = 0\right)$ the particular integrals for G, T_2, and N can easily be obtained on the basis of the law of sectorial areas for the moments G (reference 3).

2. Let $u = u(z,s)$, $v = v(z,s)$, and $w = w(z,s)$ be the components of the total displacement of some point of the middle surface of the shell taken along the generator, the tangent to the arc of the contour line, and the internal normal, respectively. For the components of the deformations corresponding (in the sense of Hooke's law) to the forces T_1, T_2, S, and G we then have the formulas

$$\left.\begin{aligned}
\epsilon_1 &= \frac{\partial u}{\partial z} \\
\gamma &= \frac{\partial u}{\partial s} + \frac{\partial v}{\partial z} \\
\epsilon_2 &= \frac{\partial v}{\partial s} - \frac{w}{R} \\
\kappa &= \frac{\partial}{\partial s}\left(\frac{v}{R} + \frac{\partial w}{\partial s}\right)
\end{aligned}\right\} \tag{2.1}$$

Eliminating the displacements we obtain the single differential equation of continuity of the deformations:

$$\frac{\partial^2}{\partial s^2}\left(R\frac{\partial^2 \epsilon_1}{\partial s^2}\right) + \frac{\partial}{\partial s}\left(\frac{1}{R}\frac{\partial \epsilon_1}{\partial s}\right) - \left[\frac{\partial^2}{\partial s^2}\left(R\frac{\partial^2 \gamma}{\partial s\, \partial z}\right) + \frac{\partial}{\partial s}\left(\frac{1}{R}\frac{\partial \gamma}{\partial z}\right)\right]$$

$$+ \frac{\partial^2}{\partial s^2}\left(R\frac{\partial^2 \epsilon_2}{\partial z^2}\right) + \frac{\partial^2 \kappa}{\partial z^2} = 0 \tag{2.2}$$

Assuming the geometrical hypotheses of the absence of deformations of transverse elongation and shear, that is, setting

$$\begin{aligned} \gamma &= 0 \\ \epsilon_2 &= 0 \end{aligned} \tag{2.3}$$

we obtain

$$\Omega\epsilon_1 + \frac{\partial^2 \kappa}{\partial z^2} = 0 \tag{2.4}$$

This is the very important equation of continuity of deformation first given in our previous papers (references 2 and 3).

Differential equation (2.4) shows that the bending deformation of an elementary transverse strip (deformation of the contour) is accompanied by the elongational deformation of the shell along the generator ("deplanation" of the cross-section).

3. Considering the shell, strengthened by ribs, as a reduced orthotropic elastic thin-walled system we represent Hooke's law under assumptions (2.3) in the following simplified form:

$$\begin{aligned} \epsilon_1 &= \frac{1}{A}\sigma \\ \kappa &= -\frac{1}{D}G \end{aligned} \tag{3.1}$$

where A is the stiffness of the shell in elongation along the generator and D is the stiffness (with account taken of the transverse ribs) in bending along the lines of the contour. If there are no ribs then evidently

$$\begin{aligned} A &= E \\ D &= \frac{E\delta^3}{12} \end{aligned} \tag{3.2}$$

where δ is the thickness of the shell, E is the modulus of elasticity.

Substituting (3.1) in equation (2.4) and combining with equation (1.2) we obtain the system of two simultaneous differential equations (proposed in reference 1):

$$\begin{aligned} \frac{\partial^2(\sigma\delta)}{\partial z^2} + \Omega G &= P \\ \Omega\sigma - \frac{A}{D}\frac{\partial^2 G}{\partial z^2} &= 0 \end{aligned} \tag{3.3}$$

The above equations have a symmetrical structure with respect to the terms with derivatives with respect to s which, as shown in references 1 and 2, is in full agreement with the fundamental theorems of the theory of elasticity. For $R = \text{constant}$ (for a circular shell) equations (3.3) will have constant coefficients.

4. Setting in the second of equations (3.3) $D = \infty$, that is, considering according to the second of equations (3.1) the contour of the cross section of the shell as rigid (nondeformable: $\kappa = 0$) we obtain

$$\frac{\partial^2}{\partial s^2}\left(R\frac{\partial^2\sigma}{\partial s^2}\right) + \frac{\partial}{\partial s}\left(\frac{1}{R}\frac{\partial\sigma}{\partial s}\right) = 0 \tag{4.1}$$

For the axial normal stresses $\sigma(z,s)$ as a function of the contour coordinate s there is thus obtained the "law of sectorial areas" which was put at the basis of our general theory of the strength, stability, and vibrations of thin-walled rods and shells of arbitrary nonsymmetrical open sections. This law, as shown in references 3 and 7, is a generalization of the plane sections hypothesis that lies at the basis of present day elementary theory of the bending of beams (the Bernoulli-Navier hypothesis). The general theory developed by us of thin-walled rods and shells with stiff profiles includes a very wide class of practically important problems on the strength, stability, and vibration of thin-walled structures (usual as well as in the elastic medium) applied in various fields of structural technology (structures, ship construction, aviation, and so forth). From this theory there arise as particular cases: the theory of the longitudinal bending of rods according to Euler, the problem of Prandtl on the stability of the plane bending of a narrow plate of rectangular section, the well-known problem of Timoshenko on the stability of the plane bending of an I-beam, and so forth.

We may note that on the basis of the theory of thin-walled rods and shells certain fundamental defects are revealed in the known works of Wagner (reference 4), Bleich (reference 5), and Pretschner on the problems of the stability of thin-walled aeronautical structures (see reference 3, page 167) and in the work of Dishinger, Ellers, Ebner, and others on the computation of cylindrical and prismatic shells on the basis of the so-called momentless theory (see reference 3, page 140).

Setting in the first of equations (3.3) $\sigma\delta = 0$, that is, assuming that the shell in its transverse sections works only in shear, as is true, for example, in the case of a crimped shell, for $P = 0$ (the homogeneous problem) we shall have

$$\frac{\partial^2}{\partial s^2}\left(R\frac{\partial^2 G}{\partial s^2}\right) + \frac{\partial}{\partial s}\left(\frac{1}{R}\frac{\partial G}{\partial s}\right) = 0 \tag{4.2}$$

The above equation agrees exactly with equation (4.1). This equation for the transverse bending moments $G = G(z,s)$ as a function of the coordinate s also gives the law of sectorial areas. On the basis of this law on introducing the four linearly independent orthogonal functions which are particular integrals of equation (4.2) there is obtained for the transverse elementary strip of the shell, in addition to the so-called elastic center in the theory of frames and which in the theory of the thin-walled rod corresponds to the center of gravity of the cross section, a new point analogous to the center of flexure.

5. Let $Z = Z(z)$ be a function of the variable z satisfying the differential equation

$$Z^{IV} - \frac{m^4}{l^4} Z = 0 \tag{5.1}$$

where l is the length of the shell in the direction of the generator, m is an arbitrary parameter. The solution of the fundamental differential equations of the shell will be sought in the form

$$\begin{aligned} G(z,s) &= G(s)Z(z) \\ u(z,s) &= u(s)Z'(z) \\ \sigma(z,s) &= \sigma(s)Z''(z) \\ S(z,s) &= S(s)Z'''(z) \end{aligned} \tag{5.2}$$

where $G(s)$, $u(s)$, $\sigma(s)$, and $S(s)$ are required functions of s. Adding to (5.2) and (5.1) the boundary conditions given in each particular case of the problem at the curvilinear edges $z = 0$ and $z = l$ of the shell, we shall have for each particular case of the boundary problem a complete system of orthogonal fundamental functions $Z_n(z)$ $(n = 1, 2, 3, \ldots, \infty)$. Each of these functions is determined by its fundamental number $m_n (n = 1, 2, 3, \ldots, \infty)$ obtained from the homogeneous boundary conditions and the differential equation (5.1). Thus, for example, in the case of a shell having at the edges $z = 0$ and $z = l$ a hinge support, the conditions for the determination of the fundamental functions $Z_n(z)$ will be $\sigma = G = 0$ for $z = 0$ and $z = l$. The fundamental functions in this case will be purely trigonometric

$$Z_n = \sin\frac{n\pi z}{l}$$

In the case of a shell both edges of which are rigidly clamped the boundary conditions at the supported edges will be

$$z = 0, \; G = u = 0$$

$$z = l, \; G = u = 0$$

For these conditions we obtain

$$Z_n(z) = (\operatorname{ch} m_n - \cos m_n)\left(\sin\frac{m_n z}{l} - \operatorname{sh}\frac{m_n z}{l}\right)$$

$$- (\operatorname{sh} m_n - \sin m_n)\left(\cos\frac{m_n z}{l} - \operatorname{ch}\frac{m_n z}{l}\right) \qquad (n = 1, 2, 3, \ldots, \infty)$$

where

$$m_1 = 4.73$$

$$m_2 = 7.853$$

$$m_n = \frac{2n + 1}{2}\pi \qquad (n > 2)$$

If one of the transverse edges $z = 0$ of the shell is hinge supported and the other $z = l$ is rigidly clamped, the boundary conditions will be

$$z = 0, \ G = \sigma = 0$$

$$z = l, \ G = u = 0$$

For these conditions the fundamental functions assume the form:

$$Z_n(z) = \sin m_n \operatorname{sh}\frac{m_n z}{l} - \cos m_n \operatorname{ch}\frac{m_n z}{l} \qquad (n = 1, 2, 3, \ldots, \infty)$$

where

$$m_1 = 3.927$$

$$m_2 = 7.068$$

$$m_n = \frac{4n + 1}{2}\pi \qquad (n > 2)$$

In a similar manner the fundamental functions $Z_n(z)$ are constructed also for other cases of boundary conditions referring to the transverse edges of the shell (one edge of the shell is hinge supported or clamped and the other free of any fixation, both edges of the shell free, and so forth).

The method of constructing the fundamental functions $Z_n(z)$ for the shells and the tables of these functions for various cases of the boundary conditions are given in our paper (reference 2). We may note that the fundamental functions $Z_n(z)$ determined by the method described above are quite the same as the functions of Rayleigh in the theory of the transverse vibrations of a homogeneous heavy beam. The deflection of the axis of the beam y, the deviation φ, the bending moment M, and the transverse force Q in our problem, as is seen from (5.2), correspond to the transverse bending moment G, the longitudinal displacement u, the normal stress σ, and the shearing force S. The function $Z_n(z)$ in this case is identical with the form $y_n(z)$ of the displacement (with deflections of the axis of the beam) corresponding to the nth frequency of its natural vibrations.

The fundamental functions $Z_n(z)$ determined by the method described above and satisfying in each particular case initially given boundary conditions for the transverse edges of the shell and the second derivatives of these functions possess the property of orthogonality:

$$\int_0^l Z_i(t)Z_k(t)dt = 0$$

$$\int_9^l Z''(t)Z_k''(t)dt = 0 \quad \text{with} \quad i \neq k \tag{5.3}$$

We now set in equations (3.3)

$$\sigma(z,s) = \sum_{1}^{\infty} \sigma_n(s) Z_n''(z)$$

$$G(z,s) = \sum_{1}^{\infty} G_n(s) Z_n(z) \tag{5.4}$$

Multiplying the first of equations (3.3) by $Z_n(z)$ and the second by $Z_n''(z)$ (n is a fixed number of the infinite series $m = 1, 2, 3, \ldots, \infty$), then taking the integrals over the entire length of the shell, and remembering (5.3), we shall have for the initial coefficients $\sigma_n(s)$ and $G_n(s)$ a system of two ordinary differential equations:

$$\lambda_n^4 \sigma_n(s) + \Omega_n G_n(s) = p_n(s)$$

$$\Omega \sigma_n(s) - \frac{A}{D} G_n(s) = 0 \tag{5.5}$$

where

$$\lambda_n^4 = \frac{m_n^4}{l^4}$$

$$p_n(s) = \int_0^l P(z,s) Zn(z) dz \tag{5.6}$$

From the second equation for a given function $P(z,s)$ depending on the external load and determined by equation (1.3) for the normalized functions $Z_n(z)$ there is determined the Fourier coefficient of the nth term of the series

$$p(z,s) = \sum_{1}^{\infty} p_n(s) Z_n(z) \tag{5.7}$$

From the system of ordinary differential equations (5.5) for fixed n (for each term of the series (5.4)) the required functions $\sigma_n(s)$ and $G_n(s)$ are determined with an accuracy up to eight arbitrary constants. These constants in each term of the series are found from the boundary conditions referring to the longitudinal edges of the shell. The number of these conditions at each point of the longitudinal edge in the computation model proposed by us is equal to four. These conditions can be given in forces or in displacements or, in the case of the mixed problem, partly in forces and partly in displacements. In this manner there is solved completely the problem of the equilibrium of a cylindrical shell of arbitrary contour for arbitrary given boundary conditions and load. For R = constant (case of the circular shell) equations (5.5), as follows from the expression (1.6) for the differential operator Ω_1, will have constant coefficients.

6. The general theory of prismatic shells consisting of a finite number of sufficiently narrow straight rectangular plates and having at the cross section arbitrarily given contours was constructed, as was shown in reference 2, on the basis of the idea of reducing the fundamental differential equations (3.3) in partial derivatives to a system of ordinary differential equations in the variable z having in the general case an eight-term structure. These equations for the required functions $\sigma_k(z)$ and $G_k(z)$ representing, respectively, the longitudinal normal stresses and the transverse bending moments referring to the kth rib of the prismatic shell (fig. 4) have the following form

$$\begin{aligned} \sum_{k=i-1}^{k=i+1} a_{ik}\sigma_k''(z) + \sum_{k=i-2}^{k=i+2} b_{ik}G_k(z) + p_i(z) &= 0 \\ \sum_{k=i-2}^{k=i+2} b_{ik}\sigma_k(z) - \sum_{k=i-1}^{k=i+1} c_{ik}G_k''(z) &= 0 \end{aligned} \tag{6.1}$$

The coefficients of these equations in the general case of thin-walled orthotropic prismatic structures (structures consisting of plates, stringers, and transverse reinforcing ribs sufficiently closely spaced along the length of the shell) are determined by the following equations

$$\left.\begin{aligned} a_{k-1,k} &= a_{k,k-1} = \frac{1}{6} F_k \\ a_{k,k} &= \frac{1}{3}\left(F_k + F_{k+1}\right) + \Delta F_k \end{aligned}\right\} \tag{6.2}$$

$$\left.\begin{aligned} \tilde{c}_{k-1,k} &= \tilde{c}_{k,k-1} = \frac{1}{6}\frac{d_k}{I_k} \\ c_{k,k} &= \frac{1}{3}\left(\frac{d_k}{I_k} + \frac{d_{k+1}}{I_{k+1}}\right) \end{aligned}\right\} \tag{6.3}$$

$$\left.\begin{aligned} b_{k-2,k} &= b_{k,k-2} = \frac{1}{d_{k-1}d_k \sin \varphi_{k-1}} \\ b_{k-1,k} &= b_{k,k-1} = -\frac{1}{d_k^2}\left(\cot \varphi_{k-1} \right. \\ &\quad \left. + \cot \varphi_k + \frac{d_k}{d_{k-1} \sin \varphi_{k-1}} + \frac{d_k}{d_{k+1} \sin \varphi_k}\right) \\ b_{k,k} &= \frac{1}{d_k^2}\left(\cot \varphi_{k-1} + \cot \varphi_k\right) + \frac{2}{d_k d_{k+1} \sin \varphi_k} \\ &\quad + \frac{1}{d^2_{k+1}}\left(\cot \varphi_k + \cot \varphi_{k+1}\right) \end{aligned}\right\} \tag{6.4}$$

In equations (6.2) F_k is the area of cross section of the kth plate between ribs $k - 1$ and k (fig. 4), and ΔF_k is the area of the cross section of the stringer placed along the kth rib and working in tension (compression) together with the plates of the shell. In equations (6.3) d_k is the width of the kth plate (the length of the side of the polygon of the cross-section located between the vertices $k - 1$ and k), and I_k is the amount of inertia per unit length of the longitudinal section of the kth plate determined in the case of the shell by the reinforcing transverse ribs with account taken of the mean moment of inertia of these ribs. In equations (6.4) φ_k is the angle between the plates k and $k + 1$ intersecting at the kth rib. In figure 4 the angles φ_{k-1}, φ_k, and φ_{k+1} are positive, and the angles φ_1 and φ_{n-1} must be considered negative.

The system of differential equations (6.1) consists of the static equations represented by the first group and corresponding to the first of equations (3.3) and the geometric equations represented by the second group and corresponding in their physical sense to the second of equations (3.3). The number of static equations (6.1) will be equal to the number of points 0, 1, 2, . . ., n of the cross section for which are determined the axial normal stresses $\sigma_k(z)$ $(k = 0, 1, . . ., n)$. The number of geometric equations (6.1) is determined by the number of required transverse bending moments referring to the ribs of the shell. For a shell with free longitudinal edges consisting of n plates the number of required functions $\sigma_k(z)$ and therefore also the number of corresponding static equations will be equal to $n + 1$. The number of required moments $G_k(z)$, however, and therefore also the number of corresponding geometric equations will be equal to $n - 3$ (the moments $G_0(z)$, $G_1(z)$, $G_{n-1}(z)$, and $G_n(z)$ in our computational model are found from the static conditions). In this case the index i in the equations of the first group (6.1) assumes $n + 1$ values and in the equations of the second group $n - 3$ values. Altogether we shall have a system of $2(n - 1)$ ordinary differential equations relative to the $2(n - 1)$ required functions, ($n + 1$ functions $\sigma_k(z)$, and $n - 3$ functions $G_k(z)$).

For the shell of closed section (fig. 5) consisting of n faces, the system (6.1) will consist of $2n$ equations of which n equations are static and n geometric. If the shell has a cylindrical hinge at any jth rib then in equations (6.1) there must be set $G_j(z) = 0$ and the corresponding jth equation of the second group rejected.

The free term $p_i(z)$ of any ith equation of the first (static) group (6.1) is determined by the equation

$$R = \frac{1}{d_{i+1}} q_{i+1}(z) - \frac{1}{d_i} q_i(z) \tag{6.5}$$

where q_i and q_{i+1} are the transverse contour unit loads acting in the planes of the faces i and $i + 1$, respectively, and directed along the increasing order number of the rib. Any external load consisting of unit forces given as functions of z and applied to the ribs of the shell can be reduced to these loads by means of expansion into series. Setting in equations (6.1)

$$\sigma_k(z) = \sum_1^{\infty} \sigma_{k,n} Z_n''(z)$$

$$G_k(z) = \sum_1^{\infty} G_{k,n} Z_n(z) \tag{6.6}$$

$$p_i(z) = \sum_1^{\infty} p_{i,n} Z_n(z)$$

where $Z_n(z)$ $(n = 1, 2, 3, \ldots, \infty)$ are fundamental functions satisfying equation (5.1) and the boundary conditions given on the transverse edges $z = 0$, $z = l$, $\sigma_{k,n}$, and $G_{k,n}$ are the required coefficients and $p_{i,n}$ the Fourier coefficients determined for the given functions $Z_n(z)$ by the formula

$$p_{i,x} = \frac{\int_0^l p_i(z) Z_n(z) dz}{\int_0^l Z_n^2(z) dz} \tag{6.7}$$

we obtain for the coefficients $\sigma_{k,n}$ and $G_{k,n}$ of the nth term of series (6.5) a system of eight-term algebraic equations:

$$
\begin{aligned}
&\lambda_n^4 \sum_{k=i-1}^{k=i+1} a_{ik}\sigma_{k,n} + \sum_{k=i-2}^{k=i+2} b_{ik}G_{k,n} + p_{i,n} = 0 \\
&\sum_{k=i-2}^{k=i+2} b_{ik}\sigma_{k,n} - \sum_{k=i-1}^{k=i+1} c_{ik}G_{k,n} = 0
\end{aligned}
\tag{6.8}
$$

The general theory of prismatic shells here presented, first given by us in the papers, references 1, 2, and 3, permitted constructing methods of solution of a number of complicated new boundary problems on the strength, stability, and vibrations of thin-walled spatial orthotropic systems for any boundary conditions given on the transverse or longitudinal edges of the shells. On the basis of this theory the cylindrical shell of arbitrary contour is considered as a prismatic shell consisting of a finite number of inscribed sides. For the computational model of the shell there is thus assumed a spatial elastic system possessing along the contour line z = constant (in the transverse section) a finite number of degrees of freedom with respect to stresses and deformations, and along the line s = constant (in the direction of the generators) an infinitely large number of degrees of freedom. Such systems are termed by us "discretely-continuous." With increase in the number of sides of the prismatic shell inscribed in (or described about) the cylindrical shell the number of equations (6.1), both geometric and static, increases. In the limit the infinite system of ordinary differential equations (6.1) goes over into the system of the two partial differential equations (3.3).

If in equations (6.1) all the coefficients c_{ik} are set equal to zero which, as is seen from (6.3), corresponds to the reinforcement of the shell by very rigid transverse ribs, then, as shown in reference 3, we shall have the law of sectorial areas for the thin-walled prismatic rods and shells possessing rigid contours.

In the same manner, setting in the equations of the first group (6.1) all the coefficients a_{ik} equal to zero, that is, assuming that in the transverse sections the shell works only in shear (for example, the shell consisting of only one crimped sheet), then for $p_i = 0$ we shall have the law of sectorial areas also for the transverse bending moments. The coefficients b_{ik} determined by equations (6.4) have a very close connection with the law of sectorial areas both for the longitudinal normal stresses $\sigma_k(z)$ as well as the transverse bending moments $G_k(z)$.

7. If the new stress function $\psi = \psi(z,s)$ is introduced by the formulas

$$\left.\begin{aligned} \sigma &= \frac{A}{D}\frac{\partial^2\psi}{\partial z^2} \\ G &= \Omega\psi \end{aligned}\right\} \tag{7.1}$$

the second of equations (3.3) is identically satisfied and the first assumes the form

$$\Omega\Omega\psi + c^2 \frac{\partial^4\psi}{\partial z^4} = P \tag{7.2}$$

where

$$c^2 = \frac{A}{D} \tag{7.3}$$

For a shell of constant thickness δ in the absence of additional transverse ribs

$$c^2 = \frac{12}{\delta^2} \tag{7.4}$$

In the case of the homogeneous problem $P = 0$, equation (7.2) breaks up into two conjugate complex equations

$$\left.\begin{aligned} \Omega\psi + ci\frac{\partial^2\psi}{\partial z^2} &= 0 \\ \Omega\bar{\psi} - ci\frac{\partial^2\bar{\psi}}{\partial z^2} &= 0 \end{aligned}\right\} \tag{7.5}$$

where i is the imaginary unit $\left(i = \sqrt{-1}\right)$.

Starting with the law of sectorial areas, that is, introducing into consideration the function $\omega = \omega(s)$ satisfying the differential equation

$$\Omega\omega = \frac{\partial^2}{\partial s^2}\left(R\frac{\partial^2 \omega}{\partial s^2}\right) + \frac{\partial}{\partial s}\left(\frac{1}{R}\frac{\partial \omega}{\partial s}\right) = 0 \tag{7.6}$$

we may represent (7.5) in the form of the equivalent integrodifferential equations:

$$\left.\begin{aligned} \psi(s,z) &= -ci\int_0^s \omega(t,s)\frac{\partial^2 \psi(t,z)}{\partial z^2}\,dt \\ \overline{\psi}(s,z) &= ci\int_0^s \omega(t,s)\frac{\partial^2 \overline{\psi}(t,z)}{\partial z^2}\,dt \end{aligned}\right\} \tag{7.7}$$

The kernel of these equations is the function $\omega(t,s)$ satisfying equation (7.6) and representing twice the area of the segment bounded by the arc $\overset{\frown}{s - t}$ and the joining chord $\overline{s - t}$. The magnitudes s and t are the coordinates of the points on the contour line (fig. 6). The function $\omega(t,s)$ will be termed the sectorial kernel.

Analogously to (7.7) equations (3.3) can be reduced to equivalent integrodifferential equations with sectorial kernel $\omega(t,s)$. The general solution of the problem (3.3) can be represented in the following form:

$$G(z,s) = G_0(z) + Y_0(z)x(s) - X_0(z)y(s) - S_0{}'(z)\omega(s)$$

$$+\int_0^s \omega(t,s)\frac{\partial^2\sigma(t,z)}{\partial z^2}\,dt + \int_0^s \omega(t,s)p(t,z)dt$$

$$\tag{7.8}$$

$$\sigma(z,s) = E\left[u_0{}'(z) - \xi_0{}''(z)x(s) - \eta_0{}''(z)y(s) - \theta_0{}''(z)\omega(s)\right]$$

$$+ \frac{A}{D}\int_0^s \omega(t,s)\frac{\partial^2 G(t,z)}{\partial z^2}\,dt$$

In these equations the following notation is assumed: $x(s)$ and $y(s)$ are the cartesian coordinates of a point on the contour line, determined by the arc s; $\omega(s)$ is the sectorial area for the point with coordinate s with the pole and origin of computation of this area at the point $s = 0$; this area is thus equal to twice the area of the segment included between the arc s and the joining chord (fig. 7); $\omega(t,s)$ is twice the area between the arc $\overset{\frown}{s - t}$ and the chord $\overline{s - t}$; $u_0(z)$, $\xi_0(z)$, $\eta_0(z)$, and $\theta_0(z)$ are, respectively, the longitudinal displacement, the displacements in the directions of the axes Ox and Oy, and the angle of twist (rotation of the tangent to the contour line) for the longitudinal edge $s = 0$ at the point determined by the coordinate z; $u_0{}'(z)$, $\xi_0{}''(z)$, $\eta_0{}''(z)$, and $\theta_0{}''(z)$ are the first and second derivatives of these variables; $G_0(z)$, $X_0(z)$, and $Y_0(z)$ are, respectively, the bending moment and the forces parallel to the axes Ox and Oy and applied at the longitudinal edge $s = 0$; $S_0(z)$ is the shearing force applied at the edge $s = 0$, and $S_0{}'(z)$ is the derivative of this force.

Of the eight magnitudes $u_0(z)$, $\xi_0(z)$, $\eta_0(z)$, $\theta_0(z)$, $G_0(z)$, $X_0(z)$, $Y_0(z)$, and $S_0(z)$ (four geometric and four static) four are usually given. For a free edge, for example, $G_0 = X_0 = Y_0 = S_0 = 0$. For an edge rigidly clamped $u_0 = \xi_0 = \eta_0 = \theta_0 = 0$. The remaining four magnitudes are obtained from the conditions given for the other edge $s = s_N$. Adding these conditions we shall have integrodifferential equations of

Fredholm with symmetric sectorial kernel. Making use of the series (5.4) and (5.7) we obtain for each nth term of these series a system of two symmetrically constructed integral equations in the unknown functions.

The method of solution here described of the problem of shells with the aid of the sectorial kernel $\omega(t,s)$ is applicable to shells having any arbitrary cross-sectional contour (for a section consisting of smooth and broken lines). Investigations show that the method of iteration for the sectorial nuclei of equations (7.8) always converges.

8. The theory of shells presented above is constructed on the basis of geometric hypotheses (2.3). If these hypotheses are not assumed, that is, if together with the deformations ϵ_1 and κ there are also taken into account the deformations ϵ_2 and γ, and if u, v, and w are eliminated from equations (2.1), we obtain as was shown previously the more general equation of continuity of deformations

$$\frac{\partial^2}{\partial s^2}\left(R\frac{\partial^2 \epsilon_1}{\partial s^2}\right) + \frac{\partial}{\partial s}\left(\frac{1}{R}\frac{\partial \epsilon_1}{\partial s}\right) - \left[\frac{\partial^2}{\partial s^2}\left(R\frac{\partial^2 \gamma}{\partial s\ \partial z}\right) + \frac{\partial}{\partial s}\left(\frac{1}{R}\frac{\partial \gamma}{\partial z}\right)\right]$$

$$+ \frac{\partial^2}{\partial s^2}\left(R\frac{\partial^2 \epsilon_2}{\partial z^2}\right) + \frac{\partial^2 \kappa}{\partial z^2} = 0 \tag{8.1}$$

This equation on the basis of Hooke's law:

$$\epsilon_1 = \frac{T_1}{A}$$

$$\epsilon_2 = \frac{T_2}{B}$$

$$\gamma = \frac{S}{C} \tag{8.2}$$

$$\kappa = -\frac{G}{D}$$

where A, B, C, and D are the coefficients of elasticity of the orthotropic shell, is reduced to the following

$$\frac{1}{A}\,\Omega T_1 - \frac{1}{C}\,\Omega_3\frac{\partial S}{\partial z} + \frac{1}{B}\,\Omega_4\frac{\partial^2 T_2}{\partial z^2} - \frac{1}{D}\,\frac{\partial^2 G}{\partial z^2} = 0 \qquad (8.3)$$

Substituting (1.5) we obtain

$$\frac{1}{A}\,\Omega\Omega\Phi + \frac{1}{C}\,\Omega_3\Omega_1\frac{\partial^2 \Phi}{\partial z^2} + \frac{1}{B}\,\Omega_4\Omega_2\frac{\partial^2 \Phi}{\partial z^2} + \frac{1}{D}\,\frac{\partial^4 \Phi}{\partial z^4} = 0 \qquad (8.4)$$

where Ω, Ω_1, Ω_2, Ω_3, and Ω_4 are the contour differential operators (in the variable s) connected with the law of sectorial areas and determined by the formulas

$$\Omega = \frac{\partial^2}{\partial s^2}\left(R\frac{\partial^2}{\partial s^2}\right) + \frac{\partial}{\partial s}\left(\frac{1}{R}\,\frac{\partial}{\partial s}\right)$$

$$\Omega_1 = \frac{\partial}{\partial s}\left(R\frac{\partial^2}{\partial s^2}\right) + \frac{1}{R}\,\frac{\partial}{\partial s}$$

$$\Omega_2 = R\frac{\partial^2}{\partial s^2} \qquad (8.5)$$

$$\Omega_3 = \frac{\partial^2}{\partial s^2}\left(R\frac{\partial}{\partial s}\right) + \frac{\partial}{\partial s}\left(\frac{1}{R}\right)$$

$$\Omega_4 = \frac{\partial^2}{\partial s^2}(R)$$

These operators for R = constant will have constant coefficients.

Equation (8.4) to which we also apply the method of separation of variables is the fundamental solving equation for the cylindrical orthotropic shell of arbitrary contour. For $R\frac{\partial^2 \Phi}{\partial s^2} = \varphi$ and $R \to \infty$ this equation goes over into the well-known equation of the fourth order for the plane problem of the theory of elasticity.

The potential energy of the shell is also represented with the aid of the stress function $\Phi = \Phi(z,s)$:

$$V = \frac{1}{2}\iint\left[\frac{1}{A}(\Omega\Phi)^2 + \frac{1}{C}\left(\Omega_1\frac{\partial\Phi}{\partial z}\right)^2 + \frac{1}{B}\left(\Omega_2\frac{\partial^2\Phi}{\partial z^2}\right)^2 + \frac{1}{D}\left(\frac{\partial^2\Phi}{\partial z^2}\right)^2\right]dz\ ds \quad (8.6)$$

It is easy to show that equation (8.4), therefore (8.2) and also equation (8.1), is the variational equation of Euler-Lagrange for the potential energy V determined by (8.6). Conversely, the principle of Castigliano, therefore also the theorem of mutual work, is a consequence of the geometric equation (8.1), the static equations (1.5), and Hooke's law (8.2). We may remark that the equations of continuity of deformations in the theory of shells were first derived by us in references 1 and 2.

For $B = C = \infty$ equation (8.6) gives the potential energy for shells working under the geometric hypotheses previously assumed.

The general technical theory of prismatic shells having in their cross sections one or several closed contours (shells of the type of the wing of an airplane) was given in reference 4. This theory is likewise based on the idea of the reduction of the shell to a discretely-continuous elastic system and the mathematical part is described by a system of symmetrically constructed ordinary differential equations obtained essentially from equations (8.4) by reducing this equation, on the basis of the principle of Lagrange, with the aid of additional physical assumptions, to ordinary differential equations analogous to equations (6.1).

The method was extended by the author also to conical shells both simple and compound with multiply connected sections. The position of a point on the middle surface of the given conical shell can be determined by two magnitudes: the coordinate z giving the distance from the vertex of the cone to the plane of the cross section of the shell passing through the given point, and the coordinate s for which may be chosen the length of the arc on the contour line of any cross section $z = c$ (for example, the base, fig. 8). For such choice of the coordinates in all the above given differential equations only the terms with the derivatives of the required functions with respect to the variable z must be changed. The fundamental equations (3.3) for the conical shell of an arbitrary section for relatively small angle (for a shell differing little from a cylindrical one) assume the form

$$\frac{z}{c}\frac{\partial}{\partial z}\left[\frac{c}{z}\frac{\partial}{\partial z}\left(\frac{z}{c}\,\sigma\delta\right)\right] + \Omega G = P$$

$$\Omega\sigma - \frac{A}{D}\frac{z}{c}\frac{\partial}{\partial z}\left[\frac{c}{z}\frac{\partial}{\partial z}\left(\frac{z}{c}\,G\right)\right] = 0 \tag{8.7}$$

where the operator Ω is determined by equation (1.6) in which $R = R(s)$ refers to the contour line $z = c$ (fig. 8). Similarly the fundamental eight-term equations (6.1) for the conical shells consisting of sufficiently long trapezoidal faces and having at the cross sections an arbitrarily given shape (fig. 9) are generalized into the following:

$$\sum_{k=i-1}^{k=i+1}{}' a_{ik}\frac{z}{c}\left[\frac{c}{z}\left(\frac{z}{c}\,\sigma\right)'\right]' + \sum_{k=i-2}^{k=i+2} b_{ik}G_k + p_i = 0$$

$$\sum_{k=i-2}^{k=i+2}{}' b_{ik}\sigma_k - \sum_{k=i-1}^{k=i+1}{}' c_{ik}\frac{z}{c}\left[\frac{c}{z}\left(\frac{z}{c}\,G\right)'\right]' = 0 \tag{8.8}$$

where the primes denote the derivatives with respect to z. The coefficients a_{ik}, b_{ik}, and c_{ik} are determined by the general equations (6.2), (6.3), and (6.4). All the geometric magnitudes entering in these equations refer to any initially chosen shape of cross section of the shell.

Equations (8.7) are also integrated by the method of separation of the variables. The fundamental functions $Z_n(z)$ $(n = 1, 2, 3, \ldots \infty)$ determined for the cylindrical shell by equation (5.1) and the boundary conditions are generalized for the case of the conical shell into the Bessel functions. For these functions equations (8.7) for each term of the series go over into ordinary differential equations (in the variable s) and equations (8.8) into a system of eight-term algebraic equations. In a similar manner there is generalized for the conical shell the theory of prismatic shells of multiply connected cross sections as given in reference 4.

9. The general problem of the strength, stability, and vibrations of thin-walled structures having the form of a sufficiently curving shell described by an arbitrarily given surface is, in our paper, reference 5, likewise reduced to a system of two simultaneous symmetrical differential equations with respect to two potential functions. This system, for shells of arbitrary cross-section shape (cylindrical, spherical, elliptical, parabolic, and so forth), has the following form:

$$\frac{1}{E\delta}\nabla_e^2\nabla_e^2\varphi - \left(H\nabla_e^2 - L\nabla_h^2\right)w = 0$$

$$\left(H\nabla_e^2 - L\nabla_h^2\right)\varphi + \frac{E\delta^3}{12\left(1 - \nu^2\right)}\nabla_e^2\nabla_e^2 w - p = 0 \tag{9.1}$$

where $\varphi = \varphi(\alpha,\beta)$ and $w = w(\alpha,\beta)$ are the required functions (α,β are the coordinates of the point of the surface in the lines of principal curvature (fig. 10)), ∇_e^2 and ∇_h^2 are the differential operators of the elliptical and hyperbolic type, respectively, for the given surface:

$$\nabla_e^2 = \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\frac{\partial}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{A}{B}\frac{\partial}{\partial\beta}\right)\right]$$

$$\nabla_h^2 = \frac{1}{AB}\left[B^2\frac{\partial}{\partial\alpha}\left(\frac{1}{AB}\frac{\partial}{\partial\alpha}\right) - A^2\frac{\partial}{\partial\beta}\left(\frac{1}{AB}\frac{\partial}{\partial\beta}\right)\right] \tag{9.2}$$

H and L are half the sum and half the difference of the principal curvatures:

$$H = \frac{1}{2}(k_1 + k_2)$$

$$L = \frac{1}{2}(k_1 - k_2) \tag{9.3}$$

The mixed auxiliary operator $H\nabla_e^2 - L\nabla_h^2$ is determined by the formula:

$$H\nabla_e^2 - L\nabla_h^2 = \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A} k_2 \frac{\partial}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{A}{B} k_1 \frac{\partial}{\partial\beta}\right)\right] \tag{9.4}$$

In addition to the fundamental equations (9.1) there are given in reference 5 the equations

$$\left.\begin{aligned} T_1 &= \frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial\varphi}{\partial\beta}\right) + \frac{1}{A^2B}\frac{\partial B}{\partial\alpha}\frac{\partial\varphi}{\partial\alpha} \\ T_2 &= \frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial\varphi}{\partial\alpha}\right) + \frac{1}{AB^2}\frac{\partial A}{\partial\beta}\frac{\partial\varphi}{\partial\beta} \\ S &= -\frac{1}{AB}\left(\frac{\partial^2\varphi}{\partial\alpha\,\partial\beta} - \frac{1}{B}\frac{\partial B}{\partial\alpha}\frac{\partial\varphi}{\partial\beta} - \frac{1}{A}\frac{\partial A}{\partial\beta}\frac{\partial\varphi}{\partial\alpha}\right) \end{aligned}\right\} \tag{9.5}$$

$$\left.\begin{aligned} \kappa_1 &= -\frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial w}{\partial\alpha}\right) - \frac{1}{AB^2}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\beta} \\ \kappa_2 &= -\frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial w}{\partial\beta}\right) - \frac{1}{A^2B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\alpha} \\ \tau &= -\frac{1}{AB}\left(\frac{\partial^2 w}{\partial\alpha\,\partial\beta} - \frac{1}{B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\beta} - \frac{1}{A}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\alpha}\right) \end{aligned}\right\} \tag{9.6}$$

Equations (9.5) representing a generalization of the well-known equations of Airy determine the axial (normal and shear) forces of the shell. These forces are expressed only in terms of the stress function. Equations (9.6) analogous in their structure to equations (9.5) refer to the deformations of bending κ_1 and κ_2 and of torsion τ of the shell. The corresponding bending and twisting moments are computed by the well-known equations of the theory of the bending of plates:

$$G_1 = -\frac{E\delta^3}{12(1-\nu^2)}\left(\kappa_1 + \nu\kappa_2\right)$$

$$G_2 = -\frac{E\delta^3}{12(1-\nu^2)}\left(\kappa_2 + \nu\kappa_1\right) \tag{9.7}$$

$$H = \frac{E\delta^3}{12(1+\nu)}\tau$$

Corresponding to (9.1), (9.5), (9.6), and (9.7) the potential energy for each shell may be expressed in terms of the two fundamental required functions $\Phi = \Phi(\alpha,\beta)$ and $w = w(\alpha,\beta)$.

The magnitudes $A = A(\alpha,\beta)$ and $B = B(\alpha,\beta)$ in equations (9.2), (9.4), (9.5), and (9.6) are the coefficients of the first quadratic form of the surface in orthogonal coordinates

$$ds^2 = A^2 d\alpha^2 + B^2 d\beta^2 \tag{9.8}$$

Equations (9.1) have the same physical sense as equations (3.3) and correspond to our mixed method. The first of equations (9.1) is obtained from the conditions of simultaneous deformations and the second from the conditions of equilibrium; $p = p(\alpha,\beta)$ is the surface load directed along the normal.

In the case of very strongly curving shells or slightly curved plates, the parameters A and B may be considered as constant magnitudes. Setting them equal to one we shall have

$$\nabla_e^2 = \frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}$$

$$\nabla_h^2 = \frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial\beta^2} \tag{9.9}$$

For shells with zero Gaussian curvature this does not hold.

The internal forces and moments are determined in terms of the potential functions φ and w by the equations

$$
\begin{aligned}
T_1 &= \frac{\partial^2 \varphi}{\partial \beta^2} \\
G_1 &= \frac{E\delta^3}{12\left(1 - \nu^2\right)}\left(\frac{\partial^2 w}{\partial \alpha^2} + \nu \frac{\partial^2 w}{\partial \beta^2}\right) \\
T_2 &= \frac{\partial^2 \varphi}{\partial \alpha^2} \\
G_2 &= \frac{E\delta^3}{12\left(1 - \nu^2\right)}\left(\frac{\partial^2 w}{\partial \beta^2} + \nu \frac{\partial^2 w}{\partial \alpha^2}\right) \\
S &= - \frac{\partial^2 \varphi}{\partial \alpha\, \partial \beta} \\
H &= - \frac{E\delta^3}{12(1 + \nu)} \frac{\partial^2 w}{\partial \alpha\, \partial \beta}
\end{aligned}
\tag{9.10}
$$

where, as in equations (9.1), $\varphi = \varphi(\alpha,\beta)$ is the stress function analogous in the plane problem of the theory of elasticity to the functions of Airy, $w = w(\alpha,\beta)$ is the normal deflection of the shell (positive if directed along the outer normal), T_1, T_2, and S are the normal and bending shearing forces (fig. 11), and G_1, G_2, and H are the bending and twisting moments (fig. 12).

The equations of the local stability of shells are given in reference 5 where are also given all the fundamental equations for the more general case of curving shells for parameters A and B determined as certain given functions of α and β.

It may be remarked that for $k_1 = k_2 = 0$, that is, in the case of the flat plate the system of equations (9.1) breaks down into the equation of Maxwell-Airy for the plane problem and the equation of Sophie Germain-Lagrange for the bending of a plate. Equations (9.1) are thus the general fundamental equations of curving shells and represent natural generalizations of the existing fundamental two-dimensional problems of the theory of elasticity referring also to the flat plate.

10. Setting in equations (9.1) $k_1 = 0$, $k_2 = \frac{1}{R}$ = constant, $A = B = R$, taking for α, β the nondimensional coordinates (the relative distances expressed in fractions of the radius R and laid off a distance αR along the generator and βR along the arc of the transverse circle), we obtain the fundamental equations for the circular cylindrical shell:

$$\frac{1}{E\delta}\nabla^2\nabla^2\varphi - R\frac{\partial^2 w}{\partial\alpha^2} = 0$$

$$R\frac{\partial^2\varphi}{\partial\alpha^2} + \frac{E\delta^3}{12(1 - \nu^2)}\nabla^2\nabla^2 w - R^4 p = 0 \tag{10.1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2} \tag{10.2}$$

Equations (10.1), on introducing the new function $\Phi = \Phi(\alpha,\beta)$ by the formulas

$$\varphi = R^2\frac{\partial^2\Phi}{\partial\alpha^2}$$

$$w = \frac{R}{E\delta}\nabla^2\nabla^2\Phi \tag{10.3}$$

are easily reduced to one equation of the eighth order:

$$c^2\nabla^2\nabla^2\nabla^2\nabla^2\Phi + \frac{\partial^4\Phi}{\partial\alpha^4} - R^2 p = 0 \tag{10.4}$$

where c^2 is a nondimensional magnitude determined by the equation

$$c^2 = \frac{\delta^2}{12(1 - \nu^2)R^2} \tag{10.5}$$

For the required internal forces, according to (9.5), (9.7), and (10.3) we obtain the general equations

$$\left.\begin{aligned}
T_1 &= \frac{\partial^4 \Phi}{\partial \alpha^2 \partial \beta^2} \\
G_1 &= Rc^2\left(\frac{\partial^2}{\partial \alpha^2} + \nu\frac{\partial^2}{\partial \beta^2}\right)\nabla^2\nabla^2\Phi \\
T_2 &= \frac{\partial^4 \Phi}{\partial \alpha^4} \\
G_2 &= Rc^2\left(\frac{\partial^2}{\partial \beta^2} + \nu\frac{\partial^2}{\partial \alpha^2}\right)\nabla^2\nabla^2\Phi \\
S &= -\frac{\partial^4 \Phi}{\partial \alpha^3 \partial \beta} \\
H &= -Rc^2(1 - \nu)\frac{\partial^2}{\partial \alpha\, \partial \beta}\nabla^2\nabla^2\Phi \\
N_1 &= -c^2\nabla^2\nabla^2\nabla^2\frac{\partial \Phi}{\partial \alpha} \\
N_2 &= -c^2\nabla^2\nabla^2\nabla^2\frac{\partial \Phi}{\partial \beta}
\end{aligned}\right\} \tag{10.6}$$

The positive directions of the forces and moments are shown in figures 13 and 14. The equations for the generalized, in the sense of Kirchhoff, transverse forces assume the form

$$N_1^* = N_1 + \frac{1}{R}\frac{\partial H}{\partial \beta} = -c^2\left[\frac{\partial^3}{\partial \alpha^3} + (2 - \nu)\frac{\partial^3}{\partial \alpha\, \partial \beta^2}\right]\nabla^2\nabla^2\Phi$$

$$N_2^* = N_2 + \frac{1}{R}\frac{\partial H}{\partial \alpha} = -c^2\left[\frac{\partial^3}{\partial \beta^3} + (2 - \nu)\frac{\partial^3}{\partial \alpha^2 \partial \beta}\right]\nabla^2\nabla^2\Phi \tag{10.7}$$

For the components of the vector of total displacement of any point of the middle surface of the shell we shall also have the equations

$$u = \frac{R}{E\delta}\left(\frac{\partial^3 \Phi}{\partial \alpha\, \partial \beta^2} - \nu\frac{\partial^3 \Phi}{\partial \alpha^3}\right)$$

$$v = -\frac{R}{E\delta}\left[\frac{\partial^3 \Phi}{\partial \beta^3} + (2 + \nu)\frac{\partial^3 \Phi}{\partial \alpha^2 \partial \beta}\right] \tag{10.8}$$

$$w = \frac{R}{E\delta}\nabla^2\nabla^2\Phi$$

where u is the longitudinal displacement (positive in the direction of increasing coordinate α), v is the displacement along the tangent to the arc of the circle of the cross section (positive in the direction of increase of the central angle β), and w is the normal displacement (positive in the direction of the outer normal).

In each particular case of the boundary problem the boundary conditions must be added to the fundamental equation (10.4). Corresponding to the physical sense of the problems here considered the number of independent conditions at each point of the edge of the shell (both the longitudinal β = constant and the transverse α = constant) must be equal to four and these conditions, depending on the character of the problem, may be either purely statical (in the forces), purely geometrical (in the displacements), or of a mixed type (part of the conditions are given in forces and part in displacements).

Since all the required forces and displacements are expressed in terms of one required function, the boundary conditions can likewise be expressed in terms of the function $\Phi = \Phi(\alpha,\beta)$.

As an example we consider a shell, the transverse edges of which $\alpha = 0$ and $\alpha = \frac{l}{R}$ (where l is the length of the shell in the direction of the generator) are hinge-connected to transverse diaphragms which are rigid in their planes. The boundary conditions in this case will be of the mixed type:

for $\alpha = 0$

$$T_1 = v = 0$$

$$G_1 = w = 0$$

for $\alpha = \frac{l}{R}$ (10.9)

$$T_1 = v = 0$$

$$G_1 = w = 0$$

The first two of these conditions in the problem of the two-dimensional stress of a rectangular plate correspond to the conditions of the freedom of motion in the axial direction of the vertical edges of a plate considered by the method of Ribière. The second two conditions in the problem of the bending of a rectangular plate correspond to the conditions of the hinge connection of a plate at its two parallel edges. These conditions are considered in the method of M. Levy. It is easy to show that the boundary conditions (10.9) expressed in terms of the function Φ will have the form for $\alpha = 0$ and $\alpha = \frac{l}{R}$:

$$\Phi = \frac{\partial^2\Phi}{\partial\alpha^2} = \frac{\partial^4\Phi}{\partial\alpha^4} = \frac{\partial^6\Phi}{\partial\alpha^6} = 0 \tag{10.10}$$

These conditions are satisfied by representing the required function in the form of a trigonometric series

$$\Phi = \sum_{1}^{\infty} \psi_n(\beta) \sin \lambda_n \alpha \tag{10.11}$$

where $\psi_n(\beta)$ are the required functions of the single variable β; λ_n is a nondimensional magnitude determined by the equation

$$\lambda_n = \frac{n\pi R}{l} \tag{10.12}$$

n is any positive number. Substituting (10.11) in equation (10.4), setting $p = 0$, we obtain for $\psi_n(\beta)$ an ordinary differential equation of the eighth order with the parameter $\lambda_n = \frac{n\pi R}{l}$:

$$\left[c^2\left(\frac{d^2}{d\beta^2} - \lambda_n{}^2\right)^4 + \lambda_n{}^4\right]\psi_n = 0 \tag{10.13}$$

The general integral of this equation is

$$\begin{aligned}\psi_n = {} & C_{1n}\psi_{1n} + C_{2n}\psi_{2n} + C_{3n}\psi_{3n} + C_{4n}\psi_{4n} + \overline{C}_{1n}\overline{\psi}_{1n} + \overline{C}_{2n}\overline{\psi}_{2n} \\ & + \overline{C}_{3n}\overline{\psi}_{3n} + \overline{C}_{4n}\overline{\psi}_{4n}\end{aligned} \tag{10.14}$$

where $C_{1n}, C_{2n}, \ldots, \overline{C}_{3n}, \overline{C}_{4n}$ are arbitrary constants, $\psi_{1n}, \psi_{2n}, \psi_{3n}, \psi_{4n}, \overline{\psi}_{1n}, \overline{\psi}_{2n}, \overline{\psi}_{3n}, \overline{\psi}_{4n}$ are particular integrals determined by the equations

$$\begin{aligned}
\psi_{1n} &= \operatorname{ch} p_n\beta \sin q_n\beta, & \overline{\psi}_{1n} &= \operatorname{ch} \overline{p}_n\beta \sin \overline{q}_n\beta \\
\psi_{2n} &= \operatorname{ch} p_n\beta \cos q_n\beta, & \overline{\psi}_{2n} &= \operatorname{ch} \overline{p}_n\beta \cos \overline{q}_n\beta \\
\psi_{3n} &= \operatorname{sh} p_n\beta \cos q_n\beta, & \overline{\psi}_{3n} &= \operatorname{sh} \overline{p}_n\beta \cos \overline{q}_n\beta \\
\psi_{4n} &= \operatorname{sh} p_n\beta \sin q_n\beta, & \overline{\psi}_{4n} &= \operatorname{sh} \overline{p}_n\beta \sin \overline{q}_n\beta
\end{aligned} \tag{10.15}$$

in which p_n, q_n, $\overline{p}_n$, $\overline{q}_n$ are magnitudes connected with the roots of the characteristic equation corresponding to (10.13) and determined for fixed value of the number n by the formulas:

$$
\begin{aligned}
p_n &= \frac{\sqrt{2\lambda_n}}{2}\sqrt{\lambda_n + \mu + \sqrt{\lambda_n^2 + 2\mu\lambda_n + 2\mu^2}} \\
\overline{p}_n &= \frac{\sqrt{2\lambda_n}}{2}\sqrt{\lambda_n - \mu + \sqrt{\lambda_n^2 - 2\mu\lambda_n + 2\mu^2}} \\
q_n &= \frac{\sqrt{2\lambda_n}}{2}\sqrt{-(\lambda_n + \mu) + \sqrt{\lambda_n^2 + 2\mu\lambda_n + 2\mu^2}} \\
\overline{q}_n &= \frac{\sqrt{2\lambda_n}}{2}\sqrt{-(\lambda_n - \mu) + \sqrt{\lambda_n^2 - 2\mu\lambda_n + 2\mu^2}}
\end{aligned}
\qquad (10.16)
$$

where

$$
\begin{aligned}
\lambda_n &= \frac{n\pi R}{l} \\
\mu &= \frac{12(1 - \nu^2)R^2}{\delta^2}
\end{aligned}
\qquad (10.17)
$$

Corresponding to the representation of the required function in the form of the series (10.11) the general integrals for the stresses and displacements of the shell can be written in the following form

$$T_{1n} = \sum_{1}^{\infty} \left[- \lambda_n^2 \frac{\partial^2 \psi_n}{\partial \beta^2} + T_{1n}^0 \right] \sin \lambda_n \alpha$$

$$T_{2n} = \sum_{1}^{\infty} \left[\lambda_n^4 \psi_n + T_{2n}^0 \right] \sin \lambda_n \alpha$$

$$S = \sum_{1}^{\infty} \left[\lambda_n^3 \frac{\partial \psi_n}{\partial \beta} + S_n^0 \right] \cos \lambda_n \alpha$$

$$G_{1n} = \sum_{1}^{\infty} \left[Rc^2 \left(\frac{d^2}{d\beta^2} - \lambda_n^2 \right)^2 \left(\nu \frac{d^2 \psi_n}{d\beta^2} - \lambda_n^2 \psi_n \right) + G_{1n}^0 \right] \sin \lambda_n \alpha$$

$$G_{2n} = \sum_{1}^{\infty} \left[Rc^2 \left(\frac{d^2}{d\beta^2} - \lambda_n^2 \right)^2 \left(\frac{d^2 \psi_n}{d\beta^2} - \nu \lambda_n^2 \psi_n \right) + G_{2n}^0 \right] \sin \lambda_n \alpha$$

$$H_n = \sum_{1}^{\infty} \left[- Rc^2 (1 - \nu) \lambda_n \left(\frac{d^2}{d\beta^2} - \lambda_n^2 \right)^2 \frac{d\psi_n}{d\beta} + H_n^0 \right] \cos \lambda_n \alpha \qquad (10.18)$$

$$N_{1n} = \sum_{1}^{\infty} \left[- c^2 \lambda_n \left(\frac{d^2}{d\beta^2} - \lambda_n^2 \right)^3 \psi_n + N_{1n}^0 \right] \cos \lambda_n \alpha$$

$$N_{2n} = \sum_{1}^{\infty} \left[- c^2 \left(\frac{d^2}{d\beta^2} - \lambda_n^2 \right)^3 \frac{d\psi_n}{d\beta} + N_{2n}^0 \right] \sin \lambda_n \alpha$$

$$u = \sum_{1}^{\infty} \left[\frac{R}{E\delta} \lambda_n \left(\frac{d^2 \psi_n}{d\beta^2} + \nu \lambda_n^2 \psi_n \right) + u_n^0 \right] \cos \lambda_n \alpha$$

$$v = \sum_{1}^{\infty} \left\{ - \frac{R}{E\delta} \left[\frac{d^3 \psi_n}{d\beta^3} - (2 + \nu) \lambda_n^2 \frac{d\psi_n}{d\beta} \right] + v_n^0 \right\} \sin \lambda_n \alpha$$

$$w = \sum_{1}^{\infty} \left[\frac{R}{E\delta} \left(\frac{d^2}{d\beta^2} - \lambda_n^2 \right)^2 \psi_n + w_n^0 \right] \sin \lambda_n \alpha$$

The derivatives of various orders entering equations (10.18) and (10.14) of the functions

$$\psi_{1n},\ \psi_{2n},\ \psi_{3n},\ \psi_{4n}$$

are computed on the basis of the recurrent formulas

$$
\begin{aligned}
\frac{d\psi_{1n}}{d\beta} &= p_n\psi_{4n} + q_n\psi_{2n}, & \frac{d^2\psi_{1n}}{d\beta^2} &= \left(p_n^{\,2} - q_n^{\,2}\right)\psi_{1n} + 2p_nq_n\psi_{3n} \\
\frac{d\psi_{2n}}{d\beta} &= p_n\psi_{3n} - q_n\psi_{1n}, & \frac{d^2\psi_{2n}}{d\beta^2} &= \left(p_n^{\,2} - q_n^{\,2}\right)\psi_{2n} - 2p_nq_n\psi_{4n} \\
\frac{d\psi_{3n}}{d\beta} &= p_n\psi_{2n} - q_n\psi_{4n}, & \frac{d^2\psi_{3n}}{d\beta^2} &= \left(p_n^{\,2} - q_n^{\,2}\right)\psi_{3n} - 2p_nq_n\psi_{1n} \\
\frac{d\psi_{4n}}{d\beta} &= p_n\psi_{1n} + q_n\psi_{3n}, & \frac{d^2\psi_{4n}}{d\beta^2} &= \left(p_n - q_n^{\,2}\right)\psi_{4n} + 2p_nq_n\psi_{2n}
\end{aligned}
\qquad (10.19)
$$

where p_n, q_n for any n are computed by formulas (10.16). Replacing in formulas (10.19) ψ_{kn} by $\overline{\psi}_{kn}$ $(k = 1, 2, 3, 4)$ the magnitudes p_n, q_n by $\overline{p}_n$, $\overline{q}_n$ determined for any n by the corresponding formulas (10.16), we obtain recurrent formulas for the derivatives of the second group of the functions $\overline{\psi}_{1n}$, $\overline{\psi}_{2n}$, $\overline{\psi}_{3n}$, and $\overline{\psi}_{4n}$.

The magnitudes $T_{1n}^{\,0}$, $T_{2n}^{\,0}$, . . ., $w_n^{\,0}$ in formulas (10.18) denote the Fourier coefficients depending in the general case on only one coordinate β and obtained as a result of representing in the form of trigonometric series the particular integrals of the fundamental nonhomogeneous differential equations referring to the given external load. The arbitrary constants C_{1n}, C_{2n}, . . ., $\overline{C}_{4n}$ the number of which in each term of the series (10.18) is equal to eight, are determined from the boundary conditions given for the straight edges of the shell (by four conditions for each edge). Thus, for example, in the case of the free (nonclamped) edges of the shell the boundary conditions assume the following form (fig. 13):

for $\beta = \pm\beta_1$

$$T_2 = S = G_2 = N_2 = 0$$

If the longitudinal edges of the shell are rigidly clamped, the constants of integration for each term of the series (10.18) are determined by the conditions:

for $\beta = \pm\beta_1$

$$u = v = w = \frac{\partial w}{\partial \beta} = 0$$

In the case of shells which are hinge-connected at the longitudinal edges so that these edges at each point have no vertical and horizontal displacements and have free motion in the longitudinal direction the boundary conditions assume the form

for $\beta = \pm\beta_1$

$$v = w = S = G_2 = 0$$

In all these cases for a load symmetrical with respect to the vertical plane the number of arbitrary constants to be determined is reduced to four. These will be the constants C_{2n}, C_{4n}, $\overline{C}_{2n}$, and $\overline{C}_{4n}$ determined for each term of the series by simultaneous solution of the linear algebraic nonhomogeneous equations. The remaining constants C_{1n}, C_{3n}, $\overline{C}_{1n}$, $\overline{C}_{3n}$ become zero.

The general solution (10.18) thus permits computing the circular cylindrical shell for any boundary conditions given on the straight edges and for any external load including a concentrated force applied at any point.

We may remark that equation (10.13) is readily integrated also in double trigonometric series (see reference 2). In this case we obtain a solution analogous to the method of Navier in the theory of the bending of plates.

It is not difficult to show that the general solution (10.18) for $R = \infty$ breaks down into two independent solutions. One of these solutions will evidently refer to the problem of the two-dimensional stress state of a plate (method of Ribiere) and the second to the problem of the bending of a plate (method of M. Levy).

Equations (10.1) obtained as particular cases of the general equations (9.1) have been applied by A. I. Lurie in the solution of the problem of the concentration of stresses of a shell near the circular opening.

11. The general problem of the local stability and vibration of curving shells, as first shown by the author in reference 5, likewise leads to the integration of a system of two homogeneous differential equations

$$\frac{1}{E\delta}\nabla_e^2\nabla_e^2\varphi - \left(H\nabla_e^2 - L\nabla_h^2\right)w = 0$$

$$\left(H\nabla_e^2 - L\nabla_h^2\right)\varphi + \frac{E\delta^3}{12(1-\nu^2)}\nabla_e^2\nabla_e^2 w$$

$$- \frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(T_1^0\frac{B}{A}\frac{\partial w}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(T_2^0\frac{A}{B}\frac{\partial w}{\partial\beta}\right)\right.$$

$$\left. + \frac{\partial}{\partial\alpha}\left(S^0\frac{\partial w}{\partial\beta}\right) + \frac{\partial}{\partial\beta}\left(S^0\frac{\partial w}{\partial\alpha}\right)\right] + \frac{\gamma\delta}{g}\frac{\partial^2 w}{\partial t^2} = 0 \qquad (11.1)$$

where T_1^0, T_2^0, S^0 are given (with an accuracy up to the parameter of the external load) internal axial forces of the shell determined by the momentless theory. For strongly curving shells and slightly curved plates equations (11.1) for $A = B \equiv 1$ assume the form

$$\frac{1}{E\delta}\nabla^2\nabla^2\varphi - \left[\frac{\partial}{\partial\alpha}\left(k_2\frac{\partial w}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(k_1\frac{\partial w}{\partial\beta}\right)\right] = 0$$

$$\frac{\partial}{\partial\alpha}\left(k_2\frac{\partial\varphi}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(k_1\frac{\partial w}{\partial\beta}\right) + \frac{E\delta^3}{12(1-\nu^2)}\nabla^2\nabla^2 w \qquad (11.2)$$

$$- \left(T_1^0\frac{\partial^2 w}{\partial\alpha^2} + 2S^0\frac{\partial^2 w}{\partial\alpha\,\partial\beta} + T_2^0\frac{\partial^2 w}{\partial\beta^2}\right) + \frac{\gamma\delta}{g}\frac{\partial^2 w}{\partial t^2} = 0$$

where $k_1 = \frac{1}{R_1}$ and $k_2 = \frac{1}{R_2}$ are the principal curvatures of the middle surface, γ is the density of the shell, δ the thickness, and g the acceleration of gravity,

$$\nabla^2 = \frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2} \tag{11.3}$$

In the case of the spherical shell $k_1 = k_2 = \frac{1}{R}$ = constant equations (11.1) will have the form

$$\frac{1}{E\delta}\nabla^2\nabla^2\varphi - \frac{1}{R}\nabla^2 w = 0$$

$$\frac{1}{R}\nabla^2\varphi + \frac{E\delta^3}{12(1 - \nu^2)}\nabla^2\nabla^2 w - \left[T_1{}^0 \frac{\partial^2 w}{\partial\alpha^2} + 2S^0 \frac{\partial^2 w}{\partial\alpha\,\partial\beta}\right. \tag{11.4}$$

$$\left. + T_2{}^0 \frac{\partial^2 w}{\partial\beta^2}\right] + \frac{\gamma\delta}{g}\frac{\partial^2 w}{\partial t^2} = 0$$

For a circular cylindrical shell for

$$k_1 = 0$$

$$k_2 = \frac{1}{R} = \text{constant}$$

$$A = B = R$$

we obtain

$$\frac{1}{E\delta}\nabla^2\nabla^2\varphi - R\frac{\partial^2 w}{\partial\alpha^2} = 0$$

$$R\frac{\partial^2\varphi}{\partial\alpha^2} + \frac{E\delta^3}{12(1 - \nu^2)}\nabla^2\nabla^2 w - R^2\left[T_1{}^0\frac{\partial^2 w}{\partial\alpha^2}\right. \qquad (11.5)$$

$$\left. + 2S^0\frac{\partial^2 w}{\partial\alpha\,\partial\beta} + T_2{}^0\frac{\partial^2 w}{\partial\beta^2}\right] + \frac{\gamma\delta}{g}R^4\frac{\partial^2 w}{\partial t^2} = 0$$

Neglecting in the above equations (11.1), (11.2), (11.4), and (11.5) the term with inertia forces, we obtain the fundamental equations of the local stability of the shells. In particular, on the basis of equations (11.4) and (11.5) we shall have the well-known solutions of a number of problems on the stability of spherical and cylindrical shells.

Setting $T_1{}^0 = T_2{}^0 = S^0 = 0$ and keeping the terms with the inertia forces we obtain the equations of the vibration of shells (cylindrical (11.5), spherical (11.4), and of arbitrary shape (11.2)). For the cylindrical shell for

$$\left.\begin{aligned} \varphi &= A \sin\lambda\alpha \cos\mu\beta \sin\omega t \\ w &= B \sin\lambda\alpha \cos\mu\beta \sin\omega t \end{aligned}\right\} \qquad (11.6)$$

$$\left.\begin{aligned} \lambda &= \frac{m\pi R}{l} \\ \mu &= \frac{n\pi}{2\beta_1} \qquad (n, m = 1, 2, 3, \ldots, \infty) \end{aligned}\right\} \qquad (11.7)$$

where R is the radius of the arc of the circle, $2\beta_1$ is the central angle, and l the length of the shell (fig. 14), we shall have on the basis of (11.5)

$$\omega_{m,n}^{2} = \frac{Eg}{\gamma R^2}\left[\frac{\delta^2}{12(1 - \nu^2)R^2}\left(\lambda^2 + \mu^2\right)^2 + \frac{\lambda^4}{\left(\lambda^2 + \mu^2\right)^2}\right] \tag{11.8}$$

From equation (11.8) are determined all the frequencies of the natural vibrations of an elastic cylindrical shell hinge-supported at all edges. For $m = n = 1$ we obtain the equation for the frequency of the fundamental tone of the vibration:

$$\omega_{1,1}^{2} = \frac{Eg}{\gamma R^2}\left\{\frac{\pi^4\delta^2}{12(1 - \nu^2)R^2}\left[\left(\frac{R}{l}\right)^2 + \left(\frac{1}{2\beta_1}\right)^2\right]^2 + \frac{R^4}{l^4\left[\left(\frac{R}{l}\right)^2 + \left(\frac{1}{2\beta_1}\right)^2\right]^2}\right\} \tag{11.9}$$

In a similar manner on the basis of more general equations is solved the problem of the vibrations of a loaded shell (a shell subjected to given stresses).

CONCLUSIONS

The technical theory of shells here presented makes possible the solution of a number of very complicated problems on the computation of thin-walled structures. Among these problems are the following:

1. The strength of thin-walled rods of arbitrary (nonsymmetrical) sections, plain or in an elastic medium. The spatial stability of the rods under the action of a longitudinal force applied at any point of the cross section.

2. The general theory of the lateral stability of thin-walled beams of arbitrary given section.

3. The general theory of the spatial flexural-torsional vibrations of thin-walled beams and similar structures (for example, girder or suspension bridges).

4. The stability of thin-walled rods having deformed contours. Central compression. Eccentric compression or tension. Pure bending. Spatial forms of the loss of stability due to the deformed contour of the rod considered as a shell in the transverse section. The interaction of the general and local stability of the rod.

5. Practical methods of computation of cylindrical and prismatic shells of various shape of cross section for arbitrarily given boundary conditions and arbitrary load.

6. Application of the theory to the computation of thin-walled structures applied in ship construction.

7. Computation of shells of the type of an airplane wing (prismatic and conical having in the cross section one or several closed contours).

8. The computation of built-up beams and columns, the elements of which are thin-walled rods.

9. The application of the theory of shells to the solution of a number of problems in the theory of elasticity. The plane problem for the rectangular region. Reduction to the one-dimensional problem by subdividing the region into sufficiently narrow rectangular strips (reference 4).

10. Application of the theory to the problem of the bending of rectangular plates and systems formed from them for arbitrarily given boundary conditions (reference 8). In this case on the basis of the method of the reduction of the plate to a discretely-continuous system we obtain also ordinary differential equations.

11. There is given a general method for the computation of beams bounded by surfaces of the first order (spherical, elliptical, parabolic, and so forth). For shells of positive Gaussian curvature the equations of the momentless theory are reduced to the equations of Cauchy-Riemann. This result first obtained by the author in the papers, references 9 and 10, permitted computation of shells of the indicated class on the basis of the theory of functions of a complex variable.

12. It is shown that the equations of momentless shells with negative Gaussian curvature bounded by surfaces of the second order reduce to the equation of the hyperbolic type with constant coefficients.

13. There is proven a theorem that the momentless shells of negative Gaussian curvature are instantaneously varying systems. For this reason the momentless theory is not applicable to such shells.

Translated by Samuel Reiss
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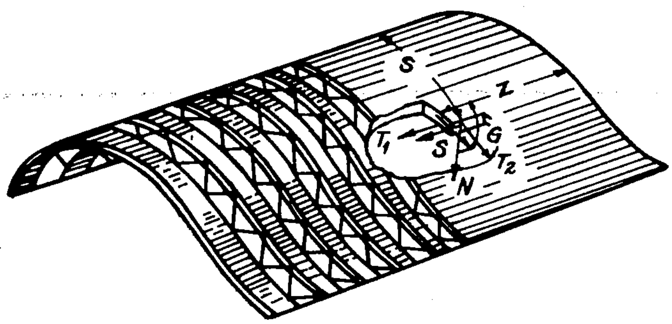
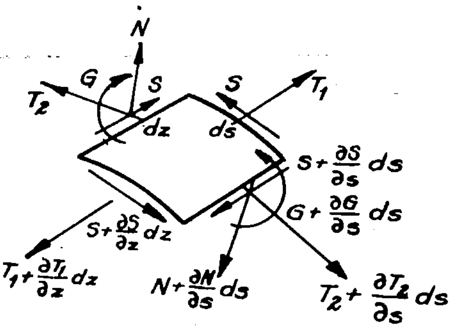


Figure 1.



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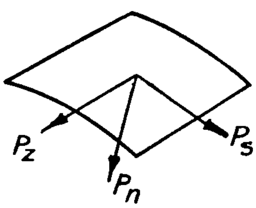

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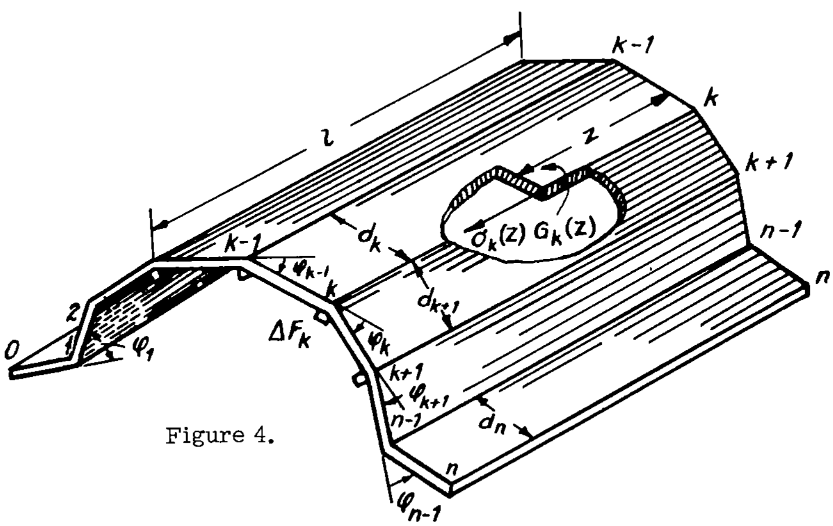

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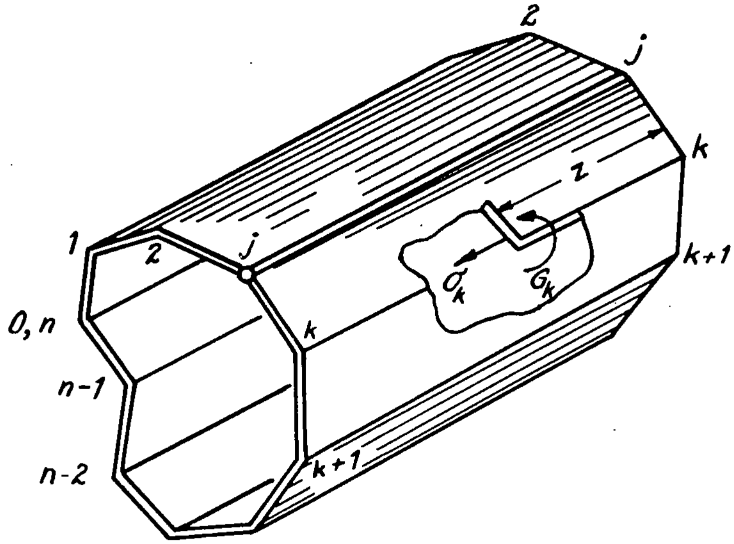

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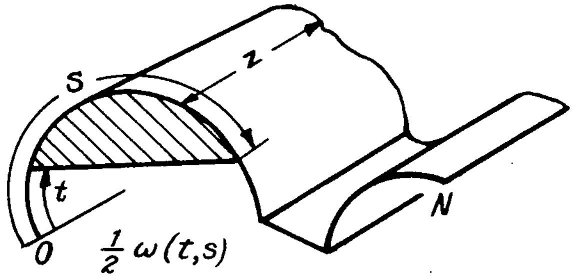

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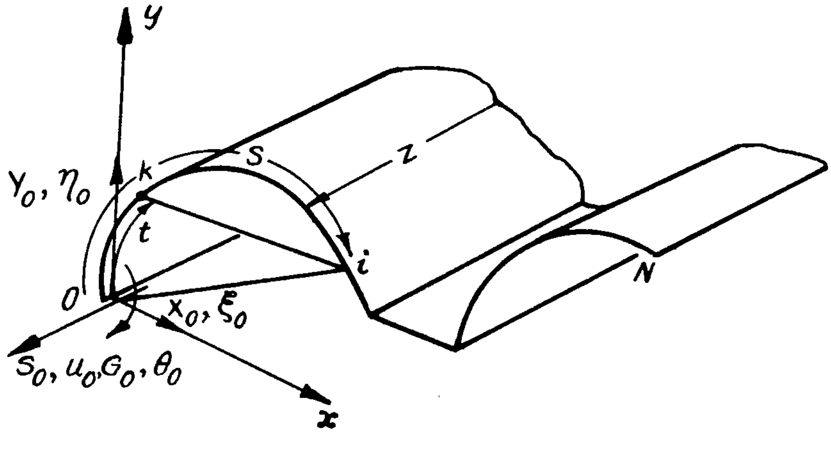

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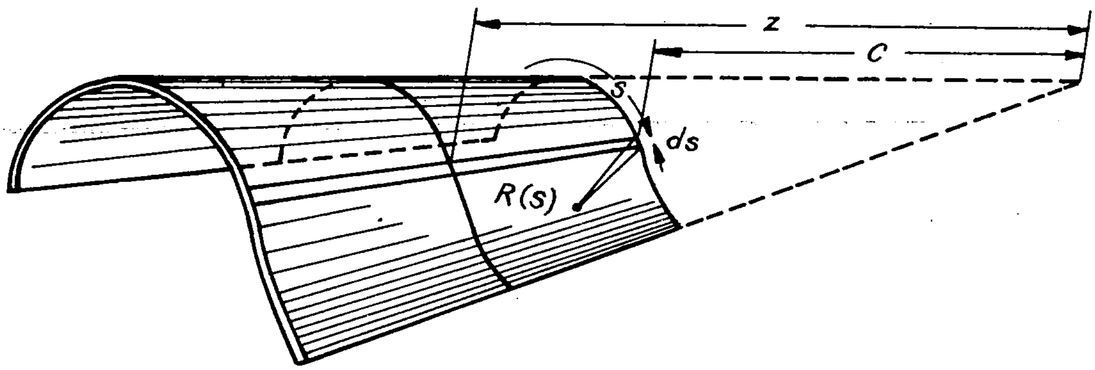

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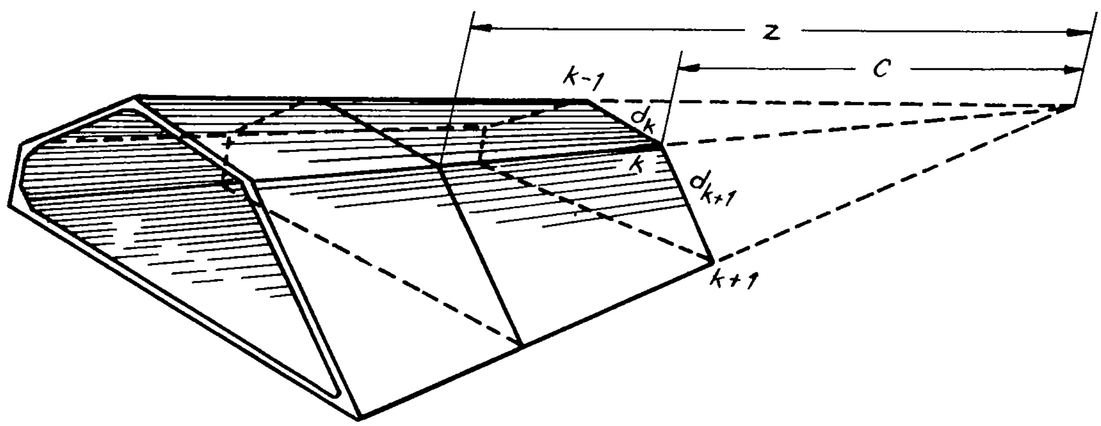

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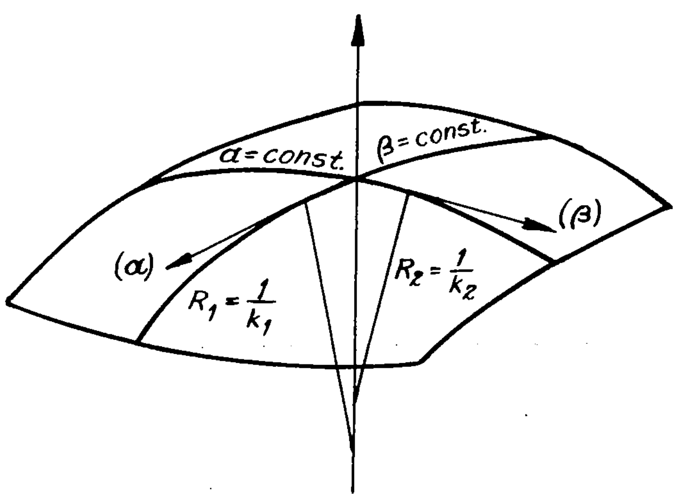

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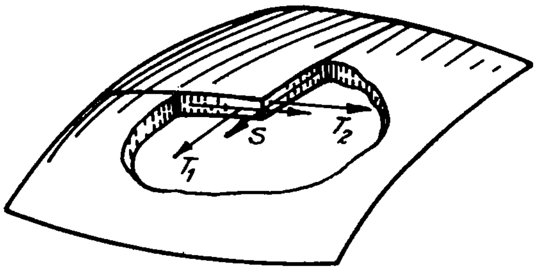
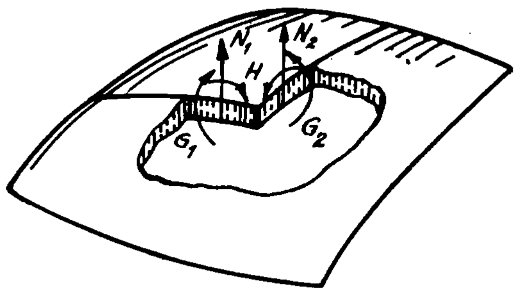


Figure 11.



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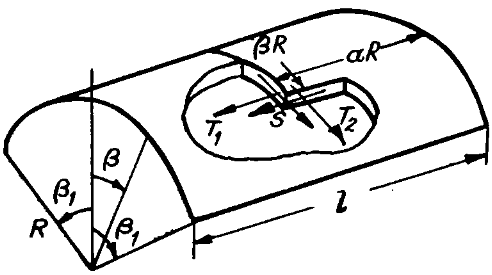

Figure 13.

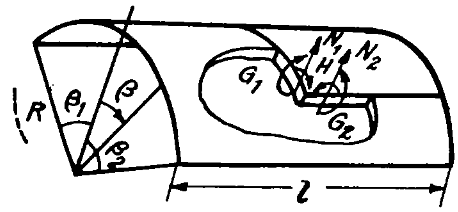

Figure 14.